PIBETA NOTE: **Understanding the Michel Positron Time Distribution**

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This note summarizes the analytical relationship between the pion stop rate and the shape of the measured time distribution of the detected Michel positrons. Limitations of this approach are discussed at the end.

1. Assumptions

(1) The accelerator period, i.e., the time difference between subsequent beam pulses is $T = 19.75\,\mathrm{ns}$.

(2) At $t = 0$ there is a π stop that opens the pion gate (πG).

(3) The detector is open (live) to events due to subsequent pulses.

(4) Pions are stopped in the target with random (Poisson) probability distribution in any beam pulse at the rate r_π.

(5) The pion and muon lifetimes are $\tau_\mu = 2197.03\,\mathrm{ns}$ and $\tau_\pi = 26.02\,\mathrm{ns}$, respectively, taken from the current Particle Properties Data Booklet.

2. Basics of sequential decay

The decay chain that leads to a Michel positron event following a pion stop proceeds through the sequence:

$$\pi \to \mu\nu \qquad \text{followed by} \qquad \mu \to e\nu\bar{\nu}\ .$$

The normalized probability distributions over time for those events are given by:

$$f_\pi(t) = \frac{1}{\tau_\pi} e^{-t/\tau_\pi} \qquad \text{and} \qquad f_\mu(t) = \frac{1}{\tau_\mu} e^{-t/\tau_\mu}\ ,$$

so that the probability distriution for sequential decay, defined only for $t \geq 0$, is given by:

$$f_{\pi\to\mu\to e}(t) = f_s(t) = \int_0^t f_\mu(t-t') f_\pi(t')\, dt' = \frac{1}{\tau_\mu - \tau_\pi}\left(e^{-t/\tau_\mu} - e^{-t/\tau_\pi}\right)\ . \qquad (1)$$

Quick inspection shows that this function is normalized, i.e., that

$$\int_0^\infty f_s(t)\, dt = 1\ .$$

3. Muon pile-up in the target

We would not need to consider anything beyond Eq. (1) if we stopped pions in the target at rates several orders of magnitude lower than the rate of muon decay ($\lambda_\mu \simeq 4.5 \times 10^5\,\mathrm{s}^{-1}$. Since this is not true, i.e., we use stopping rates comparable to and higher than the muon decay rate, we have to consider the effects of pion/muon pile-up in the target.

Given the assumptions listed above, the probability of getting a pion stop in any primary beam pulse is:

$$\eta = r_\pi \cdot T \ .$$

Thus the rate of Michel decays in the target preceding a known pion stop event at $t = 0$, due to the particles piled up in the target, is given by:

$$f_{\mathrm{pu}} = \sum_{n=1}^{\infty} \eta f_s(t + nT) \ . \tag{2}$$

This can be easily simplified by summing the infinite series. The only important caveat is that the expression in Eq. (2) is periodic and valid only in the domain $t \in [-T, 0]$. For earlier (and later) times the function is repeated with period T. Mindful of this restriction we proceed to sum the infinite series of Eq. (2).

$$\sum_{n=1}^{\infty} \eta f_s(t + nT) = \frac{\eta}{\tau_\mu - \tau_\pi} \sum_{n=1}^{\infty} \left(e^{-(t+nT)/\tau_\mu} - e^{-(t+nT)/\tau_\pi} \right) = \frac{\eta}{\tau_\mu - \tau_\pi} \left[g_\mu(t) - g_\pi(t) \right] \ .$$

The two terms must be summed independently. We do it for the first, the second term is treated in the same way:

$$g_\mu(t) = e^{-t/\tau_\mu} \sum_{n=1}^{\infty} \left(e^{-T/\tau_\mu} \right)^n = e^{-t/\tau_\mu} \left(\frac{1}{1 - e^{-T/\tau_\mu}} - 1 \right) = \frac{e^{-t/\tau_\mu}}{e^{T/\tau_\mu} - 1} \ .$$

In the same way we get:

$$g_\pi(t) = \frac{e^{-t/\tau_\pi}}{e^{T/\tau_\pi} - 1} \ .$$

It is clear that the muon term will overwhelm the pion term; however, we leave both in for completeness. Thus, we finally have:

$$f_{\mathrm{pu}} = \frac{\eta}{\tau_\mu - \tau_\pi} \left(\frac{e^{-t/\tau_\mu}}{e^{T/\tau_\mu} - 1} - \frac{e^{-t/\tau_\pi}}{e^{T/\tau_\pi} - 1} \right) \ .$$

For rates of interest in the PIBETA experiment this periodic function can be approximated with a high degree of accuracy by a constant, evaluated at $t = 0$.

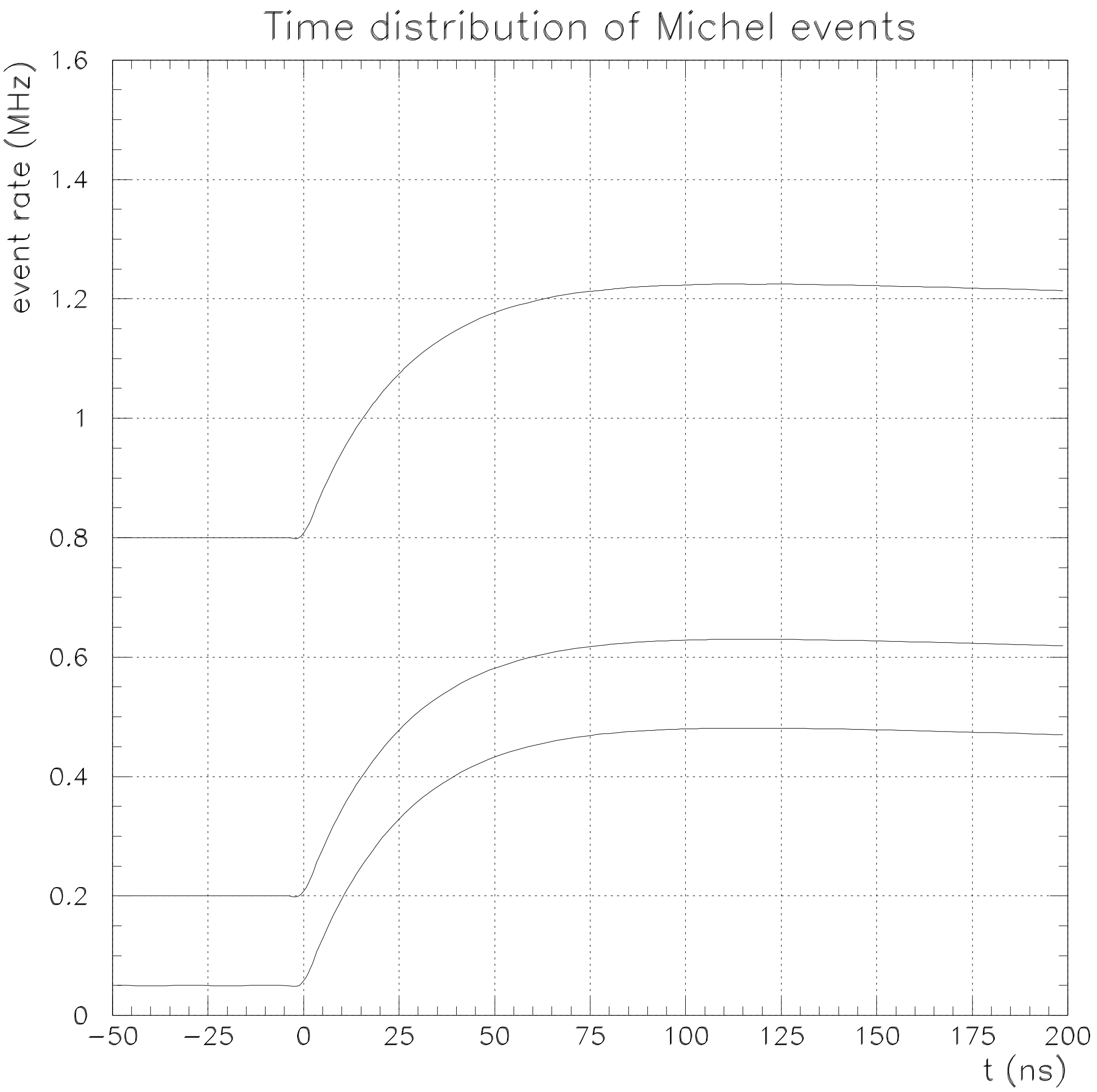

Fig. 1. Time distribution of Michel events for three pion stopping rates: 50 kHz, 200 kHz and 800 kHz. The origin of the time distribution, $t = 0$, corresponds to the time of the pion stop event responsible for opening the pion gate. The Michel event rate on the vertical axis is in MHz.

4. Full time distribution of Michel events

We can examine the shape of the predicted time distribution for the pile-up and $t = 0$ pi-stop induced Michel events. These are included in a plot in Fig. 1 of the resulting summed function $f_{\rm tot}(t) = f_s(t) + f_{\rm pu}(t)$ for three pion stopping rates: 50 kHz, 200 kHz and 800 kHz, respectively. These rates were chosen as representative of the running conditions in PIBETA during the summer/fall 1999 beam ramp-up.

An excellent independent measure of the pion stop rate is obtained by taking the ratio of the peak value of the sequential decay function $f_s^{\rm max} = f_s(t_m)$ and the pile-up value

$f_{\rm pu}(t=0)$. The relevant quantity that can be compared directly to the measured time distribution of Michel events is:

$$\xi(r_\pi) = \frac{f_s(t_m)}{f_{\rm pu}(0)} + 1 \ , \tag{3}$$

which measures the ratio of the peak of the Michel time distribution to the pile-up Michel "shelf" immediately preceding $t=0$. Setting the first derivative of $f_s(t)$ to zero we get:

$$\frac{df_s}{dt}(t_m) = \frac{1}{\tau_\mu - \tau_\pi}\left(\frac{e^{-t_m/\tau_\pi}}{\tau_\pi} - \frac{e^{-t_m/\tau_\mu}}{\tau_\mu}\right) = 0 \ ,$$

or,

$$t_m = \frac{\tau_\mu \tau_\pi}{\tau_\mu - \tau_\pi} \ln\frac{\tau_\mu}{\tau_\pi} \simeq 116.81\,\text{ns} \ .$$

Knowing all this, we can evaluate ξ from Eq. (3):

$$\xi = 1 + \frac{1}{r_\pi \cdot T}\left(e^{-t_m/\tau_\mu} - e^{-t_m/\tau_\pi}\right)\left(\frac{1}{e^{T/\tau_\mu}-1} - \frac{1}{e^{T/\tau_\pi}-1}\right)^{-1} \ , \tag{4}$$

or, inserting the above values for the muon and pion lifetimes:

$$\xi = 1 + \frac{0.93699 \cdot 109.86^{-1}}{r_\pi \cdot T} \ .$$

Using the PSI beam pulse period $T = 19.75\,$ns, we get (in the lowest order, see below for corrections):

$$\xi = 1 + \frac{0.43185}{r_\pi\,(\text{MHz})} \ . \tag{5}$$

This function is plotted in Fig. 2 for pion stop rates between 10 kHz and 2 MHz.

5. Corrections

The present report was meant as a first order analysis of (a) the pion stop rate from the time distribution of the Michel events, and (b) the overall shape of the time distribution of the Michel events.

To go beyond the first order one needs to include the following effects:

(1) Full Poisson statistics of pion stops, i.e., the exact treatment of the events with two-, three- and higher numbers of pions stopping during the same pion pulse. This affects the definition of r_π (does r_π, or does it not, count all stopped π's, be they single or accompanied by others in the beam pulse?), and leads to the substitution:

$$r_\pi \cdot T \to \frac{r'_\pi \cdot T}{1 - r'_\pi \cdot T}$$

in Eq. (4), which would require a corresponding modification of Eq. (5). These, however, are changes that depend on the precise definition and interpretation of r_π

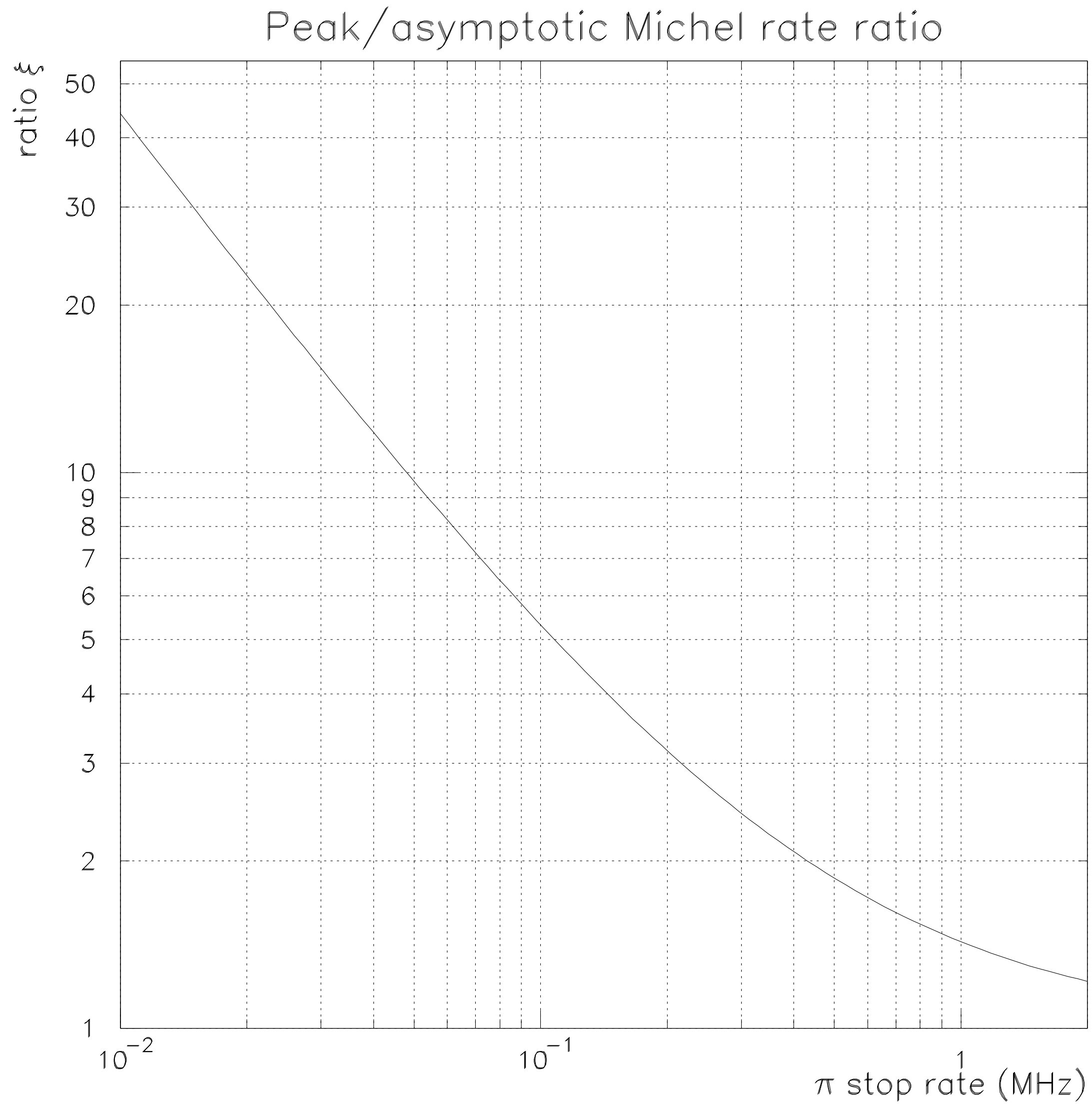

Fig. 2. Ratio ξ between the peak of the Michel time distribution function and the pre-πstop pile-up "shelf" plotted against the pion stopping rate in MHz.

and, depending on the latter, can be self-compensating. As this is a minor effect at the pion stopping rates of interest to PIBETA, it is left for the next level of refinement that will also address the issues below.

(2) Pile-up of stopped muons in the target due to *beam muons* stopping in the target, i.e., unrelated to beam pions stopped in the target. A quick analysis puts this effect down by at least two orders of magnitude, but it still needs to be evaluated, primarily in a direct experimental measurement.

(3) Effect of dead time in the detectors. This is a small effect that needs to be introduced in the analysis. That will require for us to drop the analytical approach in favor of a Monte Carlo simulation. Again, a preliminary estimate finds a small effect.

Since this work gives the first order (analytical) expressions for the quantities of interest, no effort has been made in it to propagate the uncertainties of the input parameters. All of the above points will be addressed in a forthcoming update to this analysis.